Conversions Between Parametric Curves Approximated by Fourier Series and Implicit Polynomials/Functions

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CONVERSIONS BETWEEN PARAMETRIC and IMPLICIT FORMS
Using POLAR/SPHERICAL COORDINATE REPRESENTATIONS

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ABSTRACT: Since parametric and implicit forms have complementary advantages with respect to certain geometric operations, it can be useful to convert from one form to the other. In this paper, a new method is introduced to convert between parametric and implicit forms based on polar/spherical coordinate representations.

Keywords: implicitization; parameterization; polar coordinates; spherical coordinates.

I. Introduction

The development of Computer Aided Graphics Design (CAGD) has seen two distinct approaches for representing surfaces in 3D space:
1. Parametric methods with a representation of the form \((x(u,v), y(u,v), z(u,v))\), which maps a 2D domain containing \((u,v)\) to 3D space.
2. Implicit methods that define a surface as a set of points \(\{(x,y,z) \text{ such that } F(x,y,z) = 0\}\)

The use of parametric surfaces has been quite successful for the general representation and design of free-form surfaces and remains dominant in computer graphics and geometric modeling. The implicit approach is philosophically more closely related to the concepts of Constructive Solid Geometry (CSG) and solid modeling and is receiving increased attention. Implicit surface functions naturally describe the interior of an object, whereas a parametric description of an object usually consists of piecewise surface patches. Both approaches have long lists of pros and cons [2]. Although the parametric formulation is useful for tracing, rendering and surface fitting, many operations like surface intersection desire one of the surfaces to be represented implicitly. Moreover, the implicit representation can be used for testing whether a point lies on the boundary and to represent an object as a semi-algebraic set and implicit forms are finding wider applications in computer vision, mainly in the area of object recognition and automated tolerance inspection [3,4,13-15].

Since parametric and implicit forms have complementary advantages with respect to certain geometric operations, it can be useful to convert from one form to the other. Conversion between implicit/parametric forms opens new possibilities of combining the existing vast databases of CAD models using parametric representations with the advantages of implicit polynomials for invariant object recognition.
Conversion from parametric to implicit form is known as *implicitization* and every rational surface and curve can be represented implicitly as the zero set of an irreducible\(^1\) homogeneous polynomial \(f(x,y,z,w)=0\) for surfaces, and \(f(x,y)=0\) for 2D curves [12]. Sederberg [12] applies resultants to eliminate parameters from polynomials; Hoffman [5] details the use of the Gröbner bases for the same purpose; and Hoffman [6] describes the Wu-Ritt method. The conversion from implicit to parametric form is known as *parameterization*. Parameterization is not always possible, however; for example, implicit surfaces that are defined by certain polynomials of fourth and higher degree cannot be parameterized by rational functions [9]. Conversion is always possible for nondegenerate quadrics and for cubics that have a singular point.

In this paper, a new approach based on polar coordinate representation of the coordinate system will be outlined for conversion between parametric and implicit forms. The main contribution is in the implicit/parametric conversion since no techniques are available in the literature for this type of conversion. Section 2 outlines the developed technique for converting from parametric form to implicit form and Section 3 outlines the technique for the reverse conversion. In section 4, the conversion techniques developed are applied to automated tolerance inspection of a 3D object.

### 2. Conversion from Parametric Form to Implicit Form

There are three known techniques for implicitization of parametric forms. All of these techniques reduce the problem of implicitizing rational surfaces to eliminating two variables from three parametric equations. In general, it is believed that techniques based on elimination theory can result in extraneous factors along with the implicit representation and their separation can be a difficult task. The technique developed in this section is valid only for star shaped objects.

**Theorem 1:** (Conversion in 2D) Let \(\alpha\) be a parametrized curve in \(\mathbb{R}^2\) s.t. \(\alpha: (a,b) \rightarrow \mathbb{R}^2\) with \(\alpha(t) = (h(t), g(t))\), where \((a,b)\) is an open interval in \(\mathbb{R}\). The implicit representation of this curve is given as:

\[
h^2(f^{-1}(\frac{y}{x})) + g^2(f^{-1}(\frac{y}{x})) = x^2 + y^2 \tag{1}
\]

Where \(f(t)\) is an invertible function of the form \(f(t) = \frac{g(t)}{h(t)}\).\(^1\)

**Proof:** Let \((r, \theta)\) be a polar representation of a point \(p = (x, y) \in \mathbb{R}^2\) on the curve \(\alpha\) for some \(t\). The value, \(\frac{y}{x}\), at any point \((x, y)\) on the curve and parametric representation of the value, \(\frac{g(t)}{h(t)}\), at this point is the same. Equating these two values and defining a new function of the form gives \(f(t) = \frac{g(t)}{h(t)} = \frac{y}{x}\). As the curve is star shaped, \(f(t)\) is an

---

\(^1\) If a change of variables cannot reduce the degree of the polynomial expression than it is assumed to be irreducible
invertible function. We obtain \( t = f^{-1}(\frac{Y}{X}) \). Using the standard change of variables formulas from rectangular to polar coordinates

\[
x^2 + y^2 = r^2 \quad (2) \quad x = r \cos \theta \quad (3) \quad y = r \sin \theta \quad (4)
\]

We can rewrite eqn. (2) as: \( h^2(f^{-1}(\frac{Y}{X})) + g^2(f^{-1}(\frac{Y}{X})) = r^2 = x^2 + y^2 \)

**Example 1:** Let \( h(t) = \sin t \) and \( g(t) = \sin t \cos t \)

\[
f(t) = \cos t \quad t = \cos^{-1}(\frac{Y}{X})
\]

The corresponding implicit form is: \( x^2 + y^2 = \sin^2(\cos^{-1}(\frac{Y}{X})) + \sin^2(\cos^{-1}(\frac{Y}{X})) \cos^2(\cos^{-1}(\frac{Y}{X})) \)

Simplifying this equation will yield: \( x^4 - x^2 + y^2 = 0 \)

Both parametric and implicit representations are plotted in the following figure.

![Figure 1. Plot of parametric and implicit representations](image)

The advantage of using this technique over existing techniques in the literature is that this technique is not iterative and hence is in general faster, although no systematic time comparisons are carried out. Another advantage is that there are no spurious parts in the implicit function obtained.

Theorem 1 gives the conversion of parametric functions to corresponding implicit polynomial form when \( f(t) = \frac{g(t)}{h(t)} \) is invertible. If the parametric representation of the object boundary is carried out using Fourier expansion fitting which is detailed below, the implicit function/polynomial representation of the object can be obtained more easily. This case is given in Theorem 2.

**Definition 1:** Let \( C_n \) be an \( n+1 \) dimensional vector such that

\[
C_n(2i+1) = (-1)^i a_n, \quad C_n(2i+2) = (-1)^i b_n \quad i=0,1,2,3\ldots
\]

where

\[
a_0 = \frac{1}{2N_s} \sum_{k=1}^{N_s} r(k) \quad a_n = \frac{1}{N_s} \sum_{k=1}^{N_s} r(k) \cos\left(\frac{2kn\pi}{N_s}\right)
\]
\[ b_n = \frac{1}{N_s} \sum_{k=1}^{N_s} r(k) \sin \left( \frac{2kn\pi}{N_s} \right) \]

\( a_n, b_n \) correspond to Fourier series coefficients of the parametric representation of a 2D shape \( F(r(t), t) = 0 \ t \in [0, N_s] \).

\( N_s \) is the total number of samples of the function \( r(t) \) of the contour of the 2D shape.

Define \( \otimes \) to be an operator such that
\[
\left( \sum_{i=1}^{a_{i1}} a_i \right) \otimes S_n = \sum_{i=1}^{a_{i1}} a_i S_n(i)
\]

Both the implicit polynomial representation and the parametric representation of the object is based on representing the boundary of the object in polar coordinates and expanding the radius function \( r(t) \) in terms of the angle \( \theta \) by Fourier Series. \( N_s \) is the total number of samples of the radius function \( r(t) \), from 0 to \( 2\pi \) radians. Note here that, radius function is equally sampled so \( t = \theta \).

**Theorem 2:** (Conversion from parametric representations using Fourier series fitting) An implicit function representation of any star shaped object represented by \( N^{th} \) degree Fourier series can be obtained as:

\[
(x^2 + y^2)^{1/2} = \sum_{n=0}^{N} \frac{K_n}{(x^2 + y^2)^{n/2}}
\]

where \( K_n = (x + y)^n \otimes C_n \) and \( K_0 = a_0 \)

Corresponding implicit polynomial representation is given as:

For N even:

\[
(x^2 + y^2) \left( (x^2 + y^2)^{N/2} - \sum_{m \text{ odd}}^{N} K_m (x^2 + y^2)^{N-m/2} \right)^2 = \left( \sum_{i \text{ even}}^{N} K_i (x^2 + y^2)^{N-i/2} \right)^2
\]

For N odd:

\[
\left( (x^2 + y^2)^{N+1/2} - \sum_{m \text{ odd}}^{N} K_m (x^2 + y^2)^{N-m/2} \right)^2 = (x^2 + y^2) \left( \sum_{i \text{ even}}^{N} K_i (x^2 + y^2)^{N-i/2} \right)^2
\]

**Proof:** Since radius function \( r = r(\theta) \) for a star shaped object boundary is periodic and it satisfies requirements for being a function, \( r \) can be expanded into Fourier Series with respect to index \( \theta \) as
\[ r = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta \]

Using the expansion formulas for \( \cos n\theta \) and \( \sin n\theta \) given in Appendix 1, and substituting eqns. (2) to (4) into the expanded formula, we obtain the above implicit polynomial form. 

Example 2: Two star shaped objects represented by a parametric representation using Fourier series expansion are converted to implicit polynomial form using Theorem 2. 80th degree implicit polynomial representations obtained from this conversion for Mig and Butterfly shapes are given in the following figures.

\begin{figure}[h]
  \centering
  \includegraphics[width=0.4\textwidth]{Figure2.png}
  \caption{Implicit polynomial plot of Mig}
\end{figure}

\begin{figure}[h]
  \centering
  \includegraphics[width=0.4\textwidth]{Figure3.png}
  \caption{Implicit polynomial plot of Butterfly}
\end{figure}

**Theorem 3:** (Conversion for space curves) Let \( \alpha \) be a parametrized space curve in \( \mathbb{R}^3 \) s.t. \( \alpha : (a, b) \rightarrow \mathbb{R}^3 \) with \( \alpha(t) = (h(t), g(t), k(t)) \), where \( (a, b) \) is an open interval in \( \mathbb{R} \).

The two implicit surfaces, whose intersection represents this space curve, are given as:

\begin{align}
  h^2(f_1^{-1}(\frac{y}{x})) + g^2(f_1^{-1}(\frac{y}{x})) + k^2(f_2^{-1}(\frac{z}{\sqrt{x^2 + y^2}})) &= x^2 + y^2 + z^2 \\
  h^2(f_1^{-1}(\frac{y}{x})) + g^2(f_1^{-1}(\frac{y}{x})) + k^2(f_3^{-1}(\frac{z}{y})) &= x^2 + y^2 + z^2
\end{align}

(6) (7)

Where

- \( f_1(t) \) is an invertible function of the form \( f_1(t) = \frac{g(t)}{h(t)} \)
- \( f_2(t) \) is an invertible function of the form \( f_2(t) = \frac{k(t)}{\sqrt{g^2(t) + h^2(t)}} \)
- \( f_3(t) \) is an invertible function of the form \( f_3(t) = \frac{k(t)}{g(t)} \)

**Proof:** Let \( (r, \theta, \varphi) \) be a polar representation of a point \( p = (x, y, z) \in \mathbb{R}^3 \) on the curve \( \alpha \). \( x = h(t), y = g(t), z = k(t) \) for some \( t \). \( f_1(t) \), \( f_2(t) \) and \( f_3(t) \) are obtained using equalities similar to those given in Theorem 1. These three functions are invertible since the space
curve is star shaped. We can find three different versions of \( t \) as \( t = f_1^{-1}(\frac{y}{x}) \),
\[ t = f_2^{-1}(\frac{z}{\sqrt{x^2 + y^2}}) \quad \text{and} \quad t = f_3^{-1}(\frac{z}{y}) . \]
Using the standard change of variables formulas from rectangular to spherical coordinates
\[
\begin{align*}
x^2 + y^2 + z^2 &= R^2 \quad (8) \\
x &= R \cos \theta \sin \varphi \quad (9) \\
y &= R \sin \theta \sin \varphi \quad (10) \\
z &= R \cos \varphi \quad (11)
\end{align*}
\]
We can rewrite eqn. (6) as:
\[
h^2 (f_1^{-1}(\frac{y}{x})) + g^2 (f_1^{-1}(\frac{y}{x})) + k^2 (f_2^{-1}(\frac{z}{\sqrt{x^2 + y^2}})) = R^2 = x^2 + y^2 + z^2
\]
We can rewrite eqn. (7) as:
\[
h^2 (f_1^{-1}(\frac{y}{x})) + g^2 (f_1^{-1}(\frac{y}{x})) + k^2 (f_3^{-1}(\frac{z}{y})) = R^2 = x^2 + y^2 + z^2
\]
Example 3: \( \alpha(t) = (h(t), g(t), k(t)) \) is a parametric space curve. \( h(t) = \sin t , g(t) = \cos t , k(t) = \cos^2 t . \)
\[
\begin{align*}
f_1(t) &= \frac{\cos t}{\sin t} & t &= \arccot \frac{y}{x} \\
f_2(t) &= \cos^2 t & t &= \arccos \left( \frac{z}{\sqrt{x^2 + y^2}} \right) \\
f_3(t) &= \cos t & t &= \arccos \frac{z}{y}
\end{align*}
\]
Implicit surface 1 is obtained as: \( x^2 + y^2 + z^2 = 1 + \frac{z^2}{(x^2 + y^2)^2} \)
In simplified form: \( x^2 + y^2 = 1 \)
Implicit surface 2 is obtained as: \( x^2 + y^2 + z^2 = 1 + \frac{z^4}{y^4} \)
In simplified form: \( z = \pm y^2 \) or \( z = \pm (1 - x^2) \)
Two implicit surfaces, which represent this space curve, are obtained as:
\[
\begin{align*}
x^2 + y^2 &= 1 \\
z &= 1 - x^2
\end{align*}
\]
Parametric representation of the space curve is plotted in Figure 4. Two intersecting implicit surfaces, their intersection represent the space curve, are plotted in Figure 5. It is seen that the two representations correspond to the same space curve.
Theorem 4: (Patch conversion for 3D) Let $\alpha$ be a parametrized patch in $\mathbb{R}^3$ s.t. $\alpha : [(0,2\pi),(0,2\pi)] \rightarrow \mathbb{R}^3$, with $\alpha(\theta, \varphi) = (x(\theta, \varphi), y(\theta, \varphi), z(\theta, \varphi))$. Implicit representation of this patch can be found, if $\theta$ and $\varphi$ can be obtained explicitly, in terms of $x,y,z$ from:

$$
\frac{y(\theta, \varphi)}{x(\theta, \varphi)} = \frac{\sqrt{x(\theta, \varphi)^2 + y(\theta, \varphi)^2}}{z(\theta, \varphi)} = \frac{\sqrt{x^2 + y^2}}{z}, \quad y(\theta, \varphi) = \frac{y}{z}.
$$

Implicit representation for the corresponding patch is given as:

$$
x(\theta, \varphi)^2 + y(\theta, \varphi)^2 + z(\theta, \varphi)^2 = x^2 + y^2 + z^2.
$$ (12)

Proof: The proof is the same as given for previous two theorems. The only difference in this theorem is that, an explicit slope function can not be defined here. □

Example 4: Let $\alpha(\theta, \varphi) = (x(\theta, \varphi), y(\theta, \varphi), z(\theta, \varphi))$ be patch representation of shape given in Figure 6. $x(\theta, \varphi) = \cos \theta \sin \varphi, \ y(\theta, \varphi) = \cos \theta, \ z(\theta, \varphi) = \cos \varphi$.

$\varphi$ is obtained from the equality $\frac{y}{x} = \frac{1}{\sin \varphi}$ as $\varphi = \arcsin \frac{x}{y}$

$\theta$ is obtained from the equality $\frac{\sqrt{\cos^2 \theta(1 + \sin^2 \varphi)}}{\cos \varphi} = \frac{\sqrt{x^2 + y^2}}{z}$ as

$$
\theta = \arccos \sqrt{\frac{y^2 - x^2}{z}}
$$

Replacing these values in the equality: $\cos^2 \theta(1 + \sin^2 \varphi) + \cos^2 \varphi = x^2 + y^2 + z^2$

Corresponding implicit form is obtained as: $y^2 - x^2 - y^2 z^2 = 0$
3. Conversion from Implicit Form to Parametric Form

Conversion of an implicit polynomial to a corresponding parametric form is not always possible. In this section, we will give some cases for which the implicit to parametric forms can be carried out. The conversions can be carried out exactly for the cases given in Lemmas 2-5, and approximately for more general cases.

3.1 Conversion in 2D

**Lemma 1**: A polynomial of the form

\[ \sum_{m=0}^{n} a_m x^m y^{(n-m)} + \sum_{m=0}^{k} b_m x^m y^{(k-m)} = 0 \]  

(Sum of two homogeneous polynomials) can be converted to parametric form \( \alpha(\theta) = (h(\theta), g(\theta)) \) as:

\[
\begin{align*}
    h(\theta) &= n^{-\frac{k}{2}} \left( \sum_{m=0}^{k} b_m \cos^m \theta \sin^{(k-m)} \theta \right) \\
    g(\theta) &= n^{-\frac{n}{2}} \left( \sum_{m=0}^{n} a_m \cos^m \theta \sin^{(n-m)} \theta \right)
\end{align*}
\]  

Parameter values to obtain real valued parametric form, specifies parameter range.

**Proof**: Eqn. (13) can be represented in polar coordinates as:

\[ r^n \sum_{m=0}^{n} a_m \cos^m \theta \sin^{(n-m)} \theta + r^k \sum_{m=0}^{k} b_m \cos^m \theta \sin^{(k-m)} \theta = 0 \]

If we solve the above equation for \( r \), we obtain:
\[ r^{n-k} \frac{\sum_{m=0}^{n} b_m \cos^m \theta \sin^{(k-m)} \theta}{\sum_{m=0}^{n} a_m \cos^m \theta \sin^{(n-m)} \theta} \] (16)

Using the polar to rectangular coordinate change of variable formulas \( \alpha(\theta) = (r \cos \theta, r \sin \theta) \) is obtained.

**Example 5:** Consider the implicit curve \( x^4 - x^2 + y^2 = 0 \)

\[
(r \cos \theta)^4 - (r \cos \theta)^2 + (r \sin \theta)^2 = 0
\]

\[
r^2 = \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}
\]

\[
h(\theta) = \sqrt{\cos^2 \theta - \sin^2 \theta} \quad g(\theta) = \sqrt{\cos^2 \theta - \sin^2 \theta} \frac{\sin \theta}{\cos \theta} \quad \theta \in [0,2\pi]
\]

**Example 6:** Superquadratics in 2D can be converted to parametric form by Lemma 2. A general form of a superquadratic can be given as:

\[
\left( \frac{x}{a_1} \right)^{2/\epsilon_2} + \left( \frac{y}{a_2} \right)^{2/\epsilon_1} - a = 1
\] (17)

Obtaining \( r(\theta) \) as:

\[
r(\theta) = a^{\epsilon_2/\epsilon_1} \left( \frac{\cos \theta}{a_1} \right)^{2/\epsilon_2} + \left( \frac{\sin \theta}{a_2} \right)^{2/\epsilon_2}
\]

Parametric form for the general superquadratic is then given as:

\[
h(\theta) = r(\theta) \cos \theta \quad g(\theta) = r(\theta) \sin \theta
\]

**Lemma 2:** A polynomial of the form

\[
\sum_{j=0}^{n-l} \sum_{m=0}^{n-2j} a_{jm} x^m y^{(n-j-m)} = 0
\] (18)

(Three homogeneous polynomials of consecutive degrees) can be converted to two parametric forms as:

\[
\alpha_1(\theta) = (h_1(\theta), g_1(\theta)) \quad \alpha_2(\theta) = (h_2(\theta), g_2(\theta))
\] (19) (20)

\[
h_1(\theta) = \frac{-p - \sqrt{p^2 - 4sq}}{2s} \cos \theta \quad g_1(\theta) = \frac{-p - \sqrt{p^2 - 4sq}}{2s} \sin \theta
\] (21) (22)
\[ h_2(\theta) = \frac{-p + \sqrt{p^2 - 4sq}}{2s} \cos \theta \quad (23) \quad g_2(\theta) = \frac{-p + \sqrt{p^2 - 4sq}}{2s} \sin \theta \quad (24) \]

Where

\[ s = \sum_{m=0}^{n} a_{m0} \cos^m \theta \sin^{(n-m)} \theta \quad (25) \quad p = \sum_{m=0}^{n-1} a_{m1} \cos^m \theta \sin^{(n-1-m)} \theta \quad (26) \]

\[ q = \sum_{m=0}^{n-2} a_{m2} \cos^m \theta \sin^{(n-2-m)} \theta \quad (27) \]

Parameter values to obtain real valued parametric form, specifies parameter range.

**Proof:** Eqn. (18) can be represented in polar coordinates as:

\[ \sum_{l=0}^{3} \sum_{l=0}^{m} r^{n/2-2} a_{ml} \cos^m \theta \sin^{(m-n)} \theta = 0 \]

\[ r^{n/2-2} \sum_{l=0}^{m} \sum_{m=0}^{l} a_{ml} \cos^m \theta \sin^{(m-n)} \theta = 0 \quad (28) \]

\[ sr^2 + pr + q = 0 \quad (29) \]

Solving this quadratic equation in terms of \( r \) and using polar to rectangular coordinate conversion formulas will give parametric representations of the polynomial given in eqn. (19) and eqn. (20). Closed form solution of this quadratic equation can be found in the reference [1].

**Example 7:** \[ x^2 + y^2 - 2x - 2y + 1 = 0 \]
\[ r^2 - 2r(\cos \theta + \sin \theta) + 1 = 0 \]

To find real valued roots of this equation, the following constraint has to be satisfied

\[ 4(\cos \theta + \sin \theta)^2 - 4 \geq 0 \quad \text{which is satisfied for } \theta \in \left[ 0, \frac{\pi}{2} \right] \]

The solutions of the above equation are:

\[ r_1 = \cos \theta + \sin \theta - \sqrt{\sin 2\theta} \quad r_2 = \cos \theta + \sin \theta + \sqrt{\sin 2\theta} \]

and the corresponding two parametric curves are obtained as:

\[ h_1(\theta) = r_1 \cos \theta \quad g_1(\theta) = r_1 \sin \theta \]
\[ h_2(\theta) = r_2 \cos \theta \quad g_2(\theta) = r_2 \sin \theta \quad \theta \in \left[ 0, \frac{\pi}{2} \right] \]

The implicit polynomial form of the curve given in this example is plotted in Figure 7. Corresponding two parametric forms obtained from Lemma 2 are plotted in Figures 8 and 9.
Lemma 3: A polynomial of the form $\sum_{i=0}^{3} \sum_{m=0}^{n-i} a_{m} x^{m} y^{(n-i-m)} = 0$ (30)

(Four homogeneous polynomials of consecutive degrees) can be converted to three parametric forms as:

$\alpha_{1}(t) = (h_{1}(t), g_{1}(t))$ \hspace{1cm} (31)
$\alpha_{2}(t) = (h_{2}(t), g_{2}(t))$ \hspace{1cm} (32)
$\alpha_{3}(t) = (h_{3}(t), g_{3}(t))$ \hspace{1cm} (33)

$h_{1}(\theta) = (A + B) \cos \theta$ \hspace{1cm} (34)
$g_{1}(\theta) = (A + B) \sin \theta$ \hspace{1cm} (35)

$h_{2}(\theta) = (-\frac{A+B}{2} + \frac{A-B}{2} \sqrt{-3}) \cos \theta$ \hspace{1cm} (44)
$g_{2}(\theta) = (-\frac{A+B}{2} + \frac{A-B}{2} \sqrt{-3}) \sin \theta$ \hspace{1cm} (36)

$h_{3}(\theta) = (-\frac{A+B}{2} - \frac{A-B}{2} \sqrt{-3}) \cos \theta$ \hspace{1cm} (45)
$g_{3}(\theta) = (-\frac{A+B}{2} - \frac{A-B}{2} \sqrt{-3}) \sin \theta$ \hspace{1cm} (37)

where

$A = 3 \left(\frac{b}{2} + \frac{b^{2}}{4} + \frac{a^{3}}{27}\right)$ \hspace{1cm} (38)
$B = 3 \left(\frac{b}{2} - \frac{b^{2}}{4} + \frac{a^{3}}{27}\right)$ \hspace{1cm} (39)

$a = \frac{1}{3} \left(\frac{3pQ - p^{2}}{s^{3}}\right)$ \hspace{1cm} (40)
$b = \frac{1}{27} \left(\frac{2p^{3}}{s^{3}} - 9 \frac{pq}{s^{2}} + 27 \frac{u}{s}\right)$ \hspace{1cm} (41)

$s, p, q$ \ are defined in eqns. (25) to (27).

$u = \sum_{n=0}^{n-3} a_{m3} \cos^{m} \theta \sin^{(n-3-m)} \theta$ \hspace{1cm} (42)
Parameter values to obtain real valued parametric form, specifies parameter range.

**Proof:** Eqn. (30) can be represented in polar coordinates as:

\[
\sum_{l=0}^{3} \sum_{m=0}^{n-l} r^{n-l} a_{m} \cos^{m} \theta \sin^{(n-l-m)} \theta = 0
\]

\[
r^{n-3} \sum_{l=0}^{3} \sum_{m=0}^{n-l} a_{m} \cos^{m} \theta \sin^{(n-l-m)} \theta = 0
\]  

(43)

\[sr^{3} + pr^{2} + qr + u = 0\]  

(44)

Solving this cubic equation in terms of \(r\) and using polar to rectangular coordinate conversion formulas will give parametric representations of the polynomial given in eqns. (31) to (33). Closed form solution of this cubic equation can be found in [1].

**Lemma 4:** A polynomial of the form \(\sum_{l=0}^{4} \sum_{m=0}^{n-l} a_{m} x^{m} y^{(n-l-m)} = 0\)

(45)

(Five homogeneous polynomials of consecutive degrees) can be converted to four parametric forms as:

\[\alpha_{1}(t) = (h_{1}(t), g_{1}(t))\]  

(46)

\[\alpha_{2}(t) = (h_{2}(t), g_{2}(t))\]  

(47)

\[\alpha_{3}(t) = (h_{3}(t), g_{3}(t))\]  

(48)

\[\alpha_{4}(t) = (h_{4}(t), g_{4}(t))\]  

(49)

\[h_{1}(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} + \frac{D}{2} \right) \cos \theta\]  

(50)

\[g_{1}(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} + \frac{D}{2} \right) \sin \theta\]  

(51)

\[h_{2}(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} - \frac{D}{2} \right) \cos \theta\]  

(52)

\[g_{2}(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} - \frac{D}{2} \right) \sin \theta\]  

(53)

\[h_{3}(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} + \frac{E}{2} \right) \cos \theta\]  

(54)

\[g_{3}(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} + \frac{E}{2} \right) \sin \theta\]  

(55)

\[h_{4}(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} + \frac{E}{2} \right) \cos \theta\]  

(56)

\[g_{4}(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} + \frac{E}{2} \right) \sin \theta\]  

(57)

Let \(l\) be any root of the equation \(l^{3} = \frac{q}{s} l^{2} + \left(\frac{pu}{s^{2}} - 4v\right) l^{2} - \frac{p^{2}v}{s^{3}} + 4 \frac{qv}{s^{2}} - \frac{u^{2}}{s^{2}} = 0\)

(58)

\[R = \sqrt{\frac{p}{4s} - \frac{q}{s} + l}\]  

(59)

\[D = \sqrt{\frac{3p^{2}}{4s^{2}} - R^{2} - 2 \frac{q}{s} + \frac{4pq - 8us^{2} - p^{2}}{4s^{3}R}}\]  

(60)

\[E = \sqrt{\frac{3p^{2}}{4s^{2}} - R^{2} - 2 \frac{q}{s} - \frac{4pq - 8us^{2} - p^{2}}{4s^{3}R}}\]  

(61)

\(s, p, q\) are defined in eqns. (25) to (27). \(u\) is defined in eqn. (42).
Parameter values to obtain real valued parametric form, specifies parameter range.

**Proof:** Eqn. (45) can be represented in polar coordinates as:

\[
\sum_{i=0}^{4} \sum_{m=0}^{n-i} r^{n-i} a_{ml} \cos^m \theta \sin^{(n-i-m)} \theta = 0
\]

\[
r^{n-4} \sum_{i=0}^{4} r^{4-i} \sum_{m=0}^{n-i} a_{ml} \cos^m \theta \sin^{(n-i-m)} \theta = 0
\]

\[
sr^4 + pr^3 + qr^2 + ur + v = 0
\]

Solving this quartic equation in terms of \( r \) and using polar to rectangular coordinate conversion formulas will give parametric representations of the polynomial given in eqns. (46) to (49). Solution of this quadratic equation can be found in [1].

**Theorem 5:** (Conversion in 2D) Exact parametric representation for implicit polynomials up to degree four or polynomials that are sum of up to 5 homogeneous polynomials can be obtained by representing them in polar coordinates and solving the corresponding equation to obtain \( r(\theta) \).

**Proof:** The proof follows from Lemmas 1-4.

Note that the implicit polynomials can be converted to corresponding parametric forms if the obtained equation of \( r(\theta) \) has real roots. In case the equation does not have any real roots, the implicit polynomial does not have a corresponding parametric form.

As a special case, higher order polynomials that can be factorized into terms of at most fourth order polynomials, can also be converted to parametric form term by term. An example is given below for such a situation.

**Example 8:** Consider the implicit polynomial of the form

\[
x^{22}y^{10} + x^{14}y^{18} - 10^{-5}x^{14}y^{10} - 1.9x^{20}y^{12} - 1.9x^{12}y^{20} + 19*10^{-6}x^{12}y^{12} + x^{18}y^{14} + x^{10}y^{22}
\]

\[-10^{-5}x^{10}y^{14} - 0.5x^{18}y^{10} - 0.5x^{10}y^{14} + 5*10^{-6}x^{10}y^{10} = 0
\]

The plot of this implicit polynomial is given in Figure 10.

This implicit polynomial can be factored as:

\[(x^4 - 1.9x^2y^2 + y^4 - 0.5)(x^8 + y^8 - 10^{-5})x^{10}y^{10} = 0
\]

Converting this implicit polynomial term by term:

First term: \( r^4(\cos^4 \theta - 1.9\sin^2 \theta \cos^2 \theta + \sin^4 \sin^2 \theta) = 0.5 \)
\[ h_1(\theta) = \left( \frac{0.5}{(\cos^4 \theta - 1.9 \sin^2 \theta \cos^2 \theta + \sin^4 \sin^2 \theta)} \right)^{1/4} \cos \theta \]
\[ g_1(\theta) = \left( \frac{0.5}{(\cos^4 \theta - 1.9 \sin^2 \theta \cos^2 \theta + \sin^4 \sin^2 \theta)} \right)^{1/4} \sin \theta \quad \theta \in [0, 2\pi] \]

Second term: \( r^8 (\cos^8 \theta + \sin^8 \theta) = 10^{-5} \)

\[ h_2(\theta) = \left( \frac{10^{-5}}{(\cos^8 \theta + \sin^8 \theta)} \right)^{1/8} \cos \theta \]
\[ g_2(\theta) = \left( \frac{10^{-5}}{(\cos^8 \theta + \sin^8 \theta)} \right)^{1/8} \sin \theta \quad \theta \in [0, 2\pi] \]

Third term: \( r^{10} (\cos^{10} \theta \sin^{10} \theta) = 0 \)

\[ h_3(\theta) = 0, \quad g_3(\theta) = 0 \]

![Figure 10. Shape of implicit polynomial in example 8.](image)

**Lemma 5:** (Conversion for higher order implicit polynomials) Implicit polynomials that are of order higher than 4 and cannot be written as a sum of five homogeneous polynomials can be converted to an approximate parametric form.

**Proof:** The conversion procedure is the same as in Lemmas 1-4. However, since there is no analytical solution to an equation of degree higher than 4, the solution is approximated by numerical techniques. Fourier fitting technique is then applied to obtain the approximate parametric shape.

**Example 9:** An implicit polynomial of the form \( f(x, y) = 0 \) is given to represent an object plotted in Figure 11. This implicit polynomial is converted to an approximated parametric form and both of the representations are plotted in Figure 12.

\[ f(x, y) = 4663633x^2y^2 - 51081695y^3 + 5603905960xy - 276518665x^3 + 408x^5y \]
In polar coordinate representation:

\[
f(r, \theta) = (408 \cos \theta \sin \theta + 57 \sin \theta \cos^2 \theta + 807 \cos^2 \theta - 478 \cos^4 \theta + 52 \cos^6 \theta - 500 - 467 \cos^3 \theta \sin \theta)r^6 \\
+ (31700 \sin \theta - 25461 \sin \theta \cos^2 \theta + 2434 \cos^4 \theta \sin \theta - 4156 \cos^5 \theta - 16672 \cos \theta \\
+ 27845 \cos^3 \theta)r^5 \\
+ (1253127 \cos^4 \theta + 2646954 \cos^3 \theta \sin \theta - 4326891 \cos^2 \theta - 3285437 \cos \theta \sin \theta \\
+ 4495262)r^4 \\
+ (-96384404 \cos^3 \theta - 276518665 \sin \theta + 45302709 \cos \theta + 117382579 \sin \theta \cos^3 \theta)r^3 \\
+ (-18536463436 + 5603905960 \cos \theta \sin \theta + 8788031666 \cos^2 \theta)r^2 \\
+ (72289297932 \cos \theta + 62283338323 \sin \theta)r \\
+ 30547238788903 = 0
\]

For each angle value \( \theta \), \( r(\theta) \) which is a 6\(^{th}\) order one valued polynomial, is solved to obtain the corresponding positive, real radius value and Fourier fitting is applied to these points to obtain the approximate parametric shape of the object. Fitted parametric curve is plotted on top of the original object in Figure 12.

![Figure 11. Implicit plot of an object](image1)

![Figure 12. Parametric approximation to the shape and the original object](image2)

### 3.2 Conversion in 3D

In this section, we will give results on converting an implicit polynomial in 3 variables to a corresponding parametric form. The techniques used are similar to those in 2D except that spherical coordinate representations will be used instead of polar representations. Proofs of the following Lemmas are similar to the proofs of Lemmas given for 2D, hence they are omitted.

**Lemma 6:** A polynomial of the form
\[
\sum_{m=0}^{n} \left[ a_m x^m y^{(n-m)} + b_m x^m z^{(n-m)} + c_m y^m z^{(n-m)} \right] + \sum_{r=1}^{m} \sum_{m=0}^{n} \left[ d_m x^m y^r z^{(n-m-r)} \right] \\
+ \sum_{m=0}^{k} \left[ e_m x^m y^{(k-m)} + f_m x^m z^{(k-m)} + g_m y^m z^{(k-m)} \right] + \sum_{r=1}^{k} \sum_{m=0}^{n} \left[ h_m x^m y^r z^{(k-m-r)} \right] = 0
\] (65)

can be written as a patch of the form:

\[
x(\theta, \varphi) = \\
\left[ \sum_{m=0}^{k} \left[ e_m x^m y^{(k-m)} + f_m x^m z^{(k-m)} + g_m y^m z^{(k-m)} \right] + \sum_{m=1=1}^{k} \sum_{m=0}^{k} \left[ h_m x^m y^r z^{(k-m-r)} \right] \right] \sin \varphi \cos \theta \\
- \sqrt{\sum_{m=0}^{n} \left[ a_m x^m y^{(n-m)} + b_m x^m z^{(n-m)} + c_m y^m z^{(n-m)} \right] + \sum_{m=1=1}^{n} \sum_{m=0}^{n} \left[ d_m x^m y^r z^{(n-m-r)} \right]}
\] (66)

\[
y(\theta, \varphi) = \\
\left[ \sum_{m=0}^{k} \left[ e_m x^m y^{(k-m)} + f_m x^m z^{(k-m)} + g_m y^m z^{(k-m)} \right] + \sum_{m=1=1}^{k} \sum_{m=0}^{k} \left[ h_m x^m y^r z^{(k-m-r)} \right] \right] \sin \varphi \sin \theta \\
- \sqrt{\sum_{m=0}^{n} \left[ a_m x^m y^{(n-m)} + b_m x^m z^{(n-m)} + c_m y^m z^{(n-m)} \right] + \sum_{m=1=1}^{n} \sum_{m=0}^{n} \left[ d_m x^m y^r z^{(n-m-r)} \right]}
\] (67)

\[
z(\theta, \varphi) = \\
\left[ \sum_{m=0}^{k} \left[ e_m x^m y^{(k-m)} + f_m x^m z^{(k-m)} + g_m y^m z^{(k-m)} \right] + \sum_{m=1=1}^{k} \sum_{m=0}^{k} \left[ h_m x^m y^r z^{(k-m-r)} \right] \right] \cos \varphi \\
- \sqrt{\sum_{m=0}^{n} \left[ a_m x^m y^{(n-m)} + b_m x^m z^{(n-m)} + c_m y^m z^{(n-m)} \right] + \sum_{m=1=1}^{n} \sum_{m=0}^{n} \left[ d_m x^m y^r z^{(n-m-r)} \right]}
\] (68)

Where
\[ \alpha = \sin \varphi \cos \theta \] (69)
\[ \beta = \sin \varphi \sin \theta \] (70)
\[ \gamma = \cos \varphi \] (71)

Parameter values to obtain real valued patch, specifies parameter ranges.

**Example 10:** Superquadratics in 3D can also be converted to parametric patch form by Lemma 6. A general form of a superquadratic in 3D can be given as:

\[
\left( \frac{x}{a_1} \right)^{2/\varepsilon_2} + \left( \frac{y}{a_2} \right)^{2/\varepsilon_2} + \frac{z}{a_3}^{2/\varepsilon_1} = 1
\] (72)

Obtaining \( R(\theta, \varphi) \) as:
\[ R(\theta, \varphi) = \frac{1}{\left( \frac{\cos \theta \sin \varphi}{a_1} \right)^{2/l_{y_2}} \left( \frac{\sin \theta \sin \varphi}{a_2} \right)^{2/l_{x_2}} \left( \frac{\cos \varphi}{a_2} \right)^{2/l_{x_1}} + \left( \frac{\sin \theta \sin \varphi}{a_2} \right)^{2/l_{y_2}} \left( \frac{\sin \theta \cos \varphi}{a_2} \right)^{2/l_{x_1}} } \]

Parametric patch form for the general superquadratic in 3D is given as:

\[ x(\theta, \varphi) = R(\theta, \varphi) \cos \theta \sin \varphi, \quad y(\theta, \varphi) = R(\theta, \varphi) \sin \theta \sin \varphi, \quad z(\theta, \varphi) = R(\theta, \varphi) \cos \varphi \]

**Lemma 7:** A polynomial of the form

\[
\sum_{l=0}^{n-1} \sum_{m=0}^{n-1} \left[ a_{ml} x^m y^{(n-l-m)} + b_{ml} x^m z^{(n-l-m)} + c_{ml} y^m z^{(n-l-m)} \right] + \sum_{m=1}^{n-1} \sum_{r=1}^{n-1} \left[ d_{mlr} x^m y^r z^{(n-l-m-r)} \right] = 0
\]

(73)

can be written as two patches of forms:

\[
x_1(\theta, \varphi) = -\frac{p + \sqrt{p^2 - 4sq}}{2s} \sin \varphi \cos \theta
\]

(74)

\[
y_1(\theta, \varphi) = -\frac{p + \sqrt{p^2 - 4sq}}{2s} \sin \varphi \sin \theta
\]

(75)

\[
z_1(\theta, \varphi) = -\frac{p + \sqrt{p^2 - 4sq}}{2s} \cos \varphi
\]

(76)

\[
x_2(\theta, \varphi) = -\frac{p - \sqrt{p^2 - 4sq}}{2s} \sin \varphi \cos \theta
\]

(77)

\[
y_2(\theta, \varphi) = -\frac{p - \sqrt{p^2 - 4sq}}{2s} \sin \varphi \sin \theta
\]

(78)

\[
z_2(\theta, \varphi) = -\frac{p - \sqrt{p^2 - 4sq}}{2s} \cos \varphi
\]

(79)

\[
s = \sum_{m=0}^{n-1} a_{ml} x^m y^{(n-l-m)} + b_{ml} x^m z^{(n-l-m)} + c_{ml} y^m z^{(n-l-m)} + \sum_{m=1}^{n-1} \sum_{r=1}^{n-1} d_{mlr} x^m y^r z^{(n-l-m-r)}
\]

(80)

\[
p = \sum_{m=0}^{n-1} a_{ml} x^m y^{(n-l-m)} + b_{ml} x^m z^{(n-l-m)} + c_{ml} y^m z^{(n-l-m)} + \sum_{m=1}^{n-1} \sum_{r=1}^{n-1} d_{mlr} x^m y^r z^{(n-l-m-r)}
\]

(81)

\[
q = \sum_{m=0}^{n-2} a_{ml} x^m y^{(n-2-m)} + b_{ml} x^m z^{(n-2-m)} + c_{ml} y^m z^{(n-2-m)} + \sum_{m=1}^{n-2} \sum_{r=1}^{n-2} d_{mlr} x^m y^r z^{(n-2-m-r)}
\]

(82)

\[\alpha, \beta, \gamma\] are defined in eqns. (69) to (71).

Parameter values to obtain real valued patches, specifies parameter ranges.

**Lemma 8:** A polynomial of the form

\[
\sum_{l=0}^{n-1} \sum_{m=0}^{n-1} \left[ a_{ml} x^m y^{(n-l-m)} + b_{ml} x^m z^{(n-l-m)} + c_{ml} y^m z^{(n-l-m)} \right] + \sum_{m=1}^{n-1} \sum_{r=1}^{n-1} \left[ d_{mlr} x^m y^r z^{(n-l-m-r)} \right] = 0
\]

(83)
can be written as three patches of forms:

\[ x_1(\theta, \varphi) = (A + B) \sin \varphi \cos \theta \quad (84) \]
\[ y_1(\theta, \varphi) = (A + B) \sin \varphi \sin \theta \quad (85) \]
\[ z_1(\theta, \varphi) = (A + B) \cos \varphi \quad (86) \]

\[ x_2(\theta, \varphi) = \left( -\frac{A + B}{2} + \frac{A - B}{2} \sqrt{-3} \right) \sin \varphi \cos \theta \quad (87) \]
\[ y_2(\theta, \varphi) = \left( -\frac{A + B}{2} + \frac{A - B}{2} \sqrt{-3} \right) \sin \varphi \sin \theta \quad (88) \]
\[ z_2(\theta, \varphi) = \left( -\frac{A + B}{2} + \frac{A - B}{2} \sqrt{-3} \right) \cos \varphi \quad (89) \]

\[ x_3(\theta, \varphi) = \left( -\frac{A + B}{2} - \frac{A - B}{2} \sqrt{-3} \right) \sin \varphi \cos \theta \quad (90) \]
\[ y_3(\theta, \varphi) = \left( -\frac{A + B}{2} - \frac{A - B}{2} \sqrt{-3} \right) \sin \varphi \sin \theta \quad (91) \]
\[ z_3(\theta, \varphi) = \left( -\frac{A + B}{2} - \frac{A - B}{2} \sqrt{-3} \right) \cos \varphi \quad (92) \]

Where

\[ A = \frac{1}{\sqrt{2}} \left( -\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} \right) \quad (93) \]
\[ B = \frac{1}{\sqrt{2}} \left( -\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} \right) \quad (94) \]
\[ a = \frac{1}{3} \left( 3q - \frac{p^2}{s^2} \right) \quad (95) \]
\[ b = \frac{1}{27} \left( 2p^3 - 9pq - 27u \right) \quad (96) \]

\(s, p, q\) are defined in eqns. (90) to (92).

\[ u = \sum_{m=0}^{n-1} \left[ a_{m3} \alpha^m \gamma^{(n-3-m)} + b_{m3} \alpha^m \beta^{(n-3-m)} + c_{m3} \beta^m \gamma^{(n-3-m)} \right] + \sum_{m=1}^{n-1} \sum_{l=1}^{n-1} \left[ d_{ml3} \alpha^m \beta^l \gamma^{(n-3-m-l)} \right] \quad (97) \]

\(\alpha, \beta, \gamma\) are defined in eqns. (69) to (71).

Parameter values to obtain real valued patches, specifies parameter ranges.

**Lemma 9:** A polynomial of the form

\[ \sum_{l=0}^{n-1} \sum_{m=0}^{n-1} \left[ a_{mlm} x^{l} y^{(n-l-m)} + b_{mlm} x^{(n-l-m)} + c_{mlm} y^{m} \right] + \sum_{m=1}^{n-1} \sum_{l=1}^{n-1} \left[ d_{mlm} x^{m} y^{l} z^{(n-l-m)} \right] = 0 \quad (98) \]

can be written as four patches of forms:

\[ x_1(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} \pm \frac{D}{2}) \sin \varphi \cos \theta \quad (99) \]
\[ y_1(\theta, \varphi) = \left(-\frac{p}{4s} + \frac{R}{2} + \frac{D}{2}\right) \sin \varphi \sin \theta \] (100)
\[ z_1(\theta, \varphi) = \left(-\frac{p}{4s} + \frac{R}{2} + \frac{D}{2}\right) \cos \varphi \] (101)

\[ x_2(\theta, \varphi) = \left(-\frac{p}{4s} + \frac{R}{2} - \frac{D}{2}\right) \sin \varphi \cos \theta \] (102)
\[ y_2(\theta, \varphi) = \left(-\frac{p}{4s} + \frac{R}{2} - \frac{D}{2}\right) \sin \varphi \sin \theta \] (103)
\[ z_2(\theta, \varphi) = \left(-\frac{p}{4s} + \frac{R}{2} - \frac{D}{2}\right) \cos \varphi \] (104)

\[ x_3(\theta, \varphi) = \left(-\frac{p}{4s} + \frac{R}{2} + \frac{E}{2}\right) \sin \varphi \cos \theta \] (105)
\[ y_3(\theta, \varphi) = \left(-\frac{p}{4s} + \frac{R}{2} + \frac{E}{2}\right) \sin \varphi \sin \theta \] (106)
\[ z_3(\theta, \varphi) = \left(-\frac{p}{4s} + \frac{R}{2} + \frac{E}{2}\right) \cos \varphi \] (107)

\[ x_4(\theta, \varphi) = \left(-\frac{p}{4s} + \frac{R}{2} - \frac{E}{2}\right) \sin \varphi \cos \theta \] (108)
\[ y_4(\theta, \varphi) = \left(-\frac{p}{4s} + \frac{R}{2} - \frac{E}{2}\right) \sin \varphi \sin \theta \] (109)
\[ z_4(\theta, \varphi) = \left(-\frac{p}{4s} + \frac{R}{2} - \frac{E}{2}\right) \cos \varphi \] (110)

Let \( o \) be any root of the equation
\[ o^3 - \frac{q}{s} o^2 + \left(\frac{pu}{s^2} - 4v\right) o - \frac{p^2 v}{s^2} + 4 \frac{q v}{s^2} - \frac{u^2}{s^2} = 0 \] (111)

\[ R = \sqrt{\frac{p}{4s} - \frac{q}{s} + o} \] (112)
\[ D = \sqrt{\frac{3p^2}{4s^2} - R^2 - 2 \frac{q}{s} + \frac{4pqsv - 8us^2 - p^3}{4s^3 R}} \] (113)
\[ E = \sqrt{\frac{3p^2}{4s^2} - R^2 - 2 \frac{q}{s} + \frac{4pqsv - 8us^2 - p^3}{4s^3 R}} \] (114)

\( s, p, q \) are defined in eqns. (80) to (82). \( u \) is defined in eqn. (97).

\[ v = \sum_{m=0}^{n-4} \left[ a_{m4} \alpha^m \gamma^{(n-4-m)} + b_{m4} \alpha^m \gamma^{(n-4-m)} + c_{m4} \beta^m \gamma^{(n-4-m)} \right] + \sum_{m=1}^{n-4} \sum_{j=1}^{n-4} \left[ d_{m4} \alpha^m \beta^j \gamma^{(n-4-m-j)} \right] \] (115)

\( \alpha, \beta, \gamma \) are defined in eqns. (69) to (71).

Parameter values to obtain real valued patches, specifies parameter ranges.
Theorem 6: (Conversion in 3D) Exact parametric patch representation for implicit polynomials up to degree four can be obtained by representing them in spherical coordinates and solving the corresponding equation to obtain $R(\theta, \varphi)$.

Proof: See Lemmas 6-9.

Theorem 7: (Conversion from intersecting implicit surface representation to parametric space curve representation) Let $\alpha$ be a space curve in $\mathbb{R}^3$ defined by the intersection of two implicit surfaces as $f(x, y, z) = 0, g(x, y, z) = 0$. $\alpha$ can be converted to a parametric space curve $\alpha: (a, b) \rightarrow \mathbb{R}^3$ with $\alpha(t) = (h(t), g(t), k(t))$ where $(a, b)$ is an open interval in $\mathbb{R}$, if an explicit relationship can be established between $\theta$ and $\varphi$.

Proof: The implicit polynomials $f(x, y, z) = 0, g(x, y, z) = 0$ can be represented in spherical coordinates as $f(R, \theta, \varphi) = 0, g(R, \theta, \varphi) = 0$. For each of these equalities $R$ can be obtained by solving the $R$ polynomial as in the previous Lemmas.

Let $R = F(\theta, \varphi)$ and $R = G(\theta, \varphi)$ be obtained from these two implicit polynomials. Since the space curve lies on both of the implicit surfaces, $R$ values that are same for these two implicit surfaces represent this space curve. So we can write $R = F(\theta, \varphi) = G(\theta, \varphi)$. If an explicit relationship can be established between $\theta$ and $\varphi$, as $\theta = m(\varphi)$ or $\varphi = n(\theta)$ then parametric space curve is obtained from standard change of variable formulas given in eqns.(8) to (11) as:

Assume that there is a relationship st. $\varphi = n(\theta)$.

\[ h(\theta) = R \cos(\theta \sin(n(\theta))) \quad g(\theta) = R \sin(\theta \sin(n(\theta))) \quad k(\theta) = R \cos(n(\theta)) \]

where

\[ R = F(\theta, n(\theta)) \text{ or } R = G(\theta, n(\theta)) \]

The same derivation can be done for $\theta = m(\varphi)$.

Implicit polynomials which define the space curve may yield simpler solutions as in the following example. In this example, explicit relationship is not searched, since $R$ can be found such that space curve is obtained easily.

Example 11: Let two implicit polynomials plotted in Figure 8, $x^2 + y^2 = 1$ and $z = y^5$, represent a space curve.

Converting $x^2 + y^2 = 1$ into spherical form: $R^2(\cos^2 \varphi + \sin^2 \varphi)\sin^2 \varphi = 1$

Solving the above equation for $R$: $R = \frac{1}{\sin \varphi}$

We obtain the parametric space curve $\alpha: (0, 2\pi) \rightarrow \mathbb{R}^3 \alpha(t) = (h(t), g(t), k(t))$, $h(t) = \cos t$, $g(t) = \sin t$, $k(t) = \sin^5 t$ which is plotted in red on the original implicit surfaces given in Figure 13. It is seen that $\alpha(t)$ actually lies on the intersection region.
4. Inspection by Implicit Polynomials in 3D

In this section, the theorems introduced in the previous section are applied to a real world application, automated tolerance inspection by implicit polynomials in 3D. It is assumed that, the object to be inspected is represented as an implicit polynomial form. The following theorem is given to inspect whether a given point is inside the tolerance values or not. More information about this method can be found in [15].

**Theorem 8:** (Inspection by implicit polynomial representations). Given a tolerance band ±d and an object having an implicit polynomial model \( f(x, y, z) = 0 \) to observe whether a and a point \((x_i, y_i, z_i)\) is inside tolerances, it is sufficient to check the existence of a real valued space curve defined by two implicit polynomials \( f(x, y, z) = 0 \) and

\[
(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = d^2.
\]

**Example 12:** In this example, a simple model for water pipe in implicit polynomial form, \( x^2 + y^2 = 1 \), is given. We will illustrate the technique for inspecting a single point \((0.8, 0.6, 0)\). The tolerance value is taken as ±0.1. The water pipe model and the point are plotted in Figure 14.

Space curve in implicit polynomial form for inspection is given as:

\[
x^2 + y^2 + z^2 = 0.1^2
\]

\[
(x - 0.8)^2 + (y - 0.6)^2 = 1
\]

Converting the space curve to parametric form:

\[
R = \frac{1.6 \cos \theta + 1.2 \sin \theta}{\sin \phi}
\]

or \( R = 0 \)

\[
h(\theta) = (1.6 \cos \theta + 1.2 \sin \theta) \cos \theta,
\]

\[
g(\theta) = (1.6 \cos \theta + 1.2 \sin \theta) \sin \theta
\]
\[ k(\theta) = \sqrt{0.1^2 - (1.6 \cos \theta + 1.2 \sin \theta)^2} \]

\[ \theta \in (0.6888, 0.720555) \cup (0.6888 + \pi, 0.720555 + \pi) \]

There exists at least one real \( \theta \) value such that space curve exists. So the point is inside the tolerances.

5. Conclusions

In this paper, a new method is introduced for conversions between parametric and implicit forms based on polar/spherical coordinate representations of the object boundary. The parametric to implicit conversions are carried out for star-shaped objects in 2D and 3D. Theorems given in the paper give methods in a direct way to convert parametric representations to implicit representations. The only requirement is to obtain an invertible slope function. The implicit to parametric conversions are carried out exactly for implicit polynomials up to degree four or for polynomials that are sum of up to 5 homogeneous polynomials in 2D and 3D. The conversion is done approximately for higher degree polynomials in 2D since there exists no method to solve polynomials of order greater than 4. The methods are applied in inspection procedure in the last section. Finally, the methods introduced in this paper can be used to obtain parametric and implicit forms when one is given.

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References

APPENDIX 1

We can represent $\cos n\theta$ and $\sin n\theta$ for any integer $n$, from recursive calculations starting from $\cos 2\theta$ and $\sin 2\theta$ as:

For cosine terms:
\[
\begin{align*}
\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
\cos 3\theta &= \cos^3 \theta - 3\sin^2 \theta \cos \theta \\
\cos 4\theta &= \cos^4 \theta - 6 \sin^2 \theta \cos^2 \theta + \sin^4 \theta \\
\ldots
\end{align*}
\]

For sine terms:
\[
\begin{align*}
\sin 2\theta &= 2 \sin \theta \cos \theta \\
\sin 3\theta &= 3 \sin \theta \cos^2 \theta - \sin^3 \theta \\
\sin 4\theta &= 4 \sin \theta \cos^3 \theta - 4 \cos \theta \sin^3 \theta \\
\ldots
\end{align*}
\]

APPENDIX 2

Constructing implicit polynomial form in 2D for $n = 3$.

\[
\begin{align*}
r &= a_0 + \sum_{n=1}^{3} a_n \cos n\theta + b_n \sin n\theta \\
r &= a_0 + a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta + a_3 \cos 3\theta + b_3 \sin 3\theta \\
r &= a_0 + a_1 \cos \theta + b_1 \sin \theta + a_2 (\cos^2 \theta - \sin^2 \theta) + b_2 2\sin \theta \cos \theta + a_3 (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) \\
&\quad\quad + b_3 (3 \sin \theta \cos^2 \theta - \sin^3 \theta) \\
r &= a_0 + \frac{a_1 x + b_1 y}{r} + \frac{a_2 x^2 + b_2 2xy - a_2 y^2}{r^2} + \frac{a_3 x^3 + b_3 3x^2y - a_3 3xy^2 - b_3 y^3}{r^3} \\
r &= a_0 + \frac{(x + y) \otimes C_1}{r} + \frac{(x + y)^2 \otimes C_2}{r^2} + \frac{(x + y)^3 \otimes C_3}{r^3}
\end{align*}
\]

Final implicit function form is obtained as:
\[
(x^2 + y^2)^{1/2} = \sum_{n=0}^{3} \frac{K_n}{(x^2 + y^2)^{n/2}}
\]

Where $K_n = (x + y)^n \otimes C_n$ and $K_0 = a_0$

The above equation can be converted to implicit polynomial form as:
\[
\begin{align*}
(x^2 + y^2)^{1/2} &= K_0 + \frac{K_1}{(x^2 + y^2)^{1/2}} + \frac{K_2}{(x^2 + y^2)^{3/2}} + \frac{K_3}{(x^2 + y^2)^{5/2}} \\
\left((x^2 + y^2)^2 - K_1(x^2 + y^2) - K_2\right)^{1/2} &= (x^2 + y^2) \left(K_0(x^2 + y^2) + K_2\right)^{1/2}
\end{align*}
\]