CONVERSIONS BETWEEN PARAMETRIC AND IMPLICIT FORMS USING POLAR/SHPHERICAL COORDINATE REPRESENTATIONS

Cem Ünsalan
Aytül Erçil

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Fen Bilimleri Enstitüsü
Institute for Graduate Studies in Science and Engineering

Boğaziçi University, Bebek, Istanbul, Turkey

Boğaziçi Araştırmaları deneme niteliğinde olup, bilimsel tartışmaya katkı amacıyla yayınlan-dıklarından, yazar(lar)ın yazılı izin olmaksızın kendilerine atıfta bulunulamaz.

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ABSTRACT: Conversions between parametric and implicit forms of curves are considered. A new method is introduced to convert parametric form of a curve to a corresponding implicit form. Also a new method is introduced to convert implicit form of a curve to a corresponding parametric form. Polar/spherical coordinate representations are used in both methods.

Keywords: Curves in plane, curves in space, parametric form, implicit form, polar coordinates, spherical coordinates.

1. Introduction

The development of Computer Aided Graphics Design (CAGD) has seen two distinct approaches for representing surfaces in 3D space:

1. Parametric methods with a representation of the form \( (x(u,v),y(u,v), z(u,v)) \), which maps a 2D domain containing \((u,v)\) to 3D space.
2. Implicit methods that define a surface as a set of points \( \{(x,y,z) \text{ such that } F(x,y,z) = 0 \} \)

The use of parametric surfaces has been quite successful for the general representation and design of free-form surfaces and remains dominant in computer graphics and geometric modeling. The implicit approach is philosophically more closely related to the concepts of Constructive Solid Geometry (CSG) and solid modeling and is receiving increased attention. Implicit surface functions naturally describe the interior of an object, whereas a parametric description of an object usually consists of piecewise surface patches. Both approaches have long lists of pros and cons [2]. Although the parametric formulation is useful for tracing, rendering and surface fitting, many operations like surface intersection desire one of the surfaces to be represented implicitly. Moreover, the implicit representation can be used for testing whether a point lies on the boundary and to represent an object as a semi-algebraic set and implicit forms are finding wider applications in computer vision, mainly in the area of object recognition [3,4,13,14].

Since parametric and implicit forms have complementary advantages with respect to certain geometric operations, it can be useful to convert from one form to the other. Conversion from parametric to implicit form is known as *implicitization* and every rational surface and curve can be represented implicitly as the zero set of an irreducible homogeneous polynomial \( f(x,y,z,w)=0 \) for surfaces, and \( f(x,y)=0 \) for 2D curves [12].

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1 If a change of variables cannot reduce the degree of the polynomial expression than it is assumed to be irreducible
Sederberg [12] applies resultants to eliminate parameters from polynomials; Hoffman [5] details the use of the Gröbner bases for the same purpose; and Hoffman [6] describes the Wu-Ritt method. The conversion from implicit to parametric form is known as parameterization. Parameterization is not always possible, however; for example, implicit surfaces that are defined by certain polynomials of fourth and higher degree cannot be parameterized by rational functions [9]. Conversion is always possible for nondegenerate quadrics and for cubics that have a singular point.

In this paper a new method based on polar coordinate representations in 2D and spherical coordinate representations in 3D are introduced to convert parametric form to a corresponding implicit form. Parametric form is represented in polar or spherical coordinates and conversion is achieved. Also a new method is introduced to covert implicit form to a corresponding parametric form. In this method polar and spherical coordinate representations are used.

The layout of the paper is as follows. In section two, conversion from parametric to implicit form is given for 2D and 3D curves. In section three, conversion from implicit to parametric form is given for 2D and 3D surfaces. Conclusions are given in section four.

2. Conversion from Parametric Form to Implicit Form

There are three known techniques for implicitization of parametric forms. All of these techniques reduce the problem of implicitizing rational surfaces to eliminating two variables from three parametric equations. In general, it is believed that techniques based on elimination theory can result in extraneous factors along with the implicit representation and their separation can be a difficult task. Furthermore, these algorithms are not able to implicitize parametric surfaces like bicubic patches in a reasonable amount of time and space [9].

A new method based on polar representation for 2D curves and spherical representation for 3D surfaces is introduced to convert a parametric form to a corresponding implicit form in this section. The technique is illustrated with several examples.

2.1 Conversion in 2D

In this section a method based on polar coordinate transformations will be used to carry out implicit to parametric conversion in 2D. The method is originated from defining slope function of a curve in parametric form and equating this form to its implicit correspondence. Using standard change of variable formulas from rectangular to polar coordinate representations, implicitization is simply achieved. The following theorem gives mathematical basis for this conversion method.

**Theorem 1:** Let \( \alpha \) be a parametrized curve in \( R^n \) s.t. \( \alpha: (a, b) \rightarrow R^n \) with \( \alpha(t) = (h(t), g(t)) \), where \( (a, b) \) is an open interval in \( R \). The implicit representation of this curve is given as:

\[
h^2 \left( f^{-1} \left( \frac{y}{x} \right) \right) + g^2 \left( f^{-1} \left( \frac{y}{x} \right) \right) = x^2 + y^2
\]

Where \( f(t) \) is an injective function of the form \( f(t) = \frac{g(t)}{h(t)} \)
Proof: Let \((r, \theta)\) be a polar representation of a point \(p = (x, y) \in \mathbb{R}^2\) on the curve \(\alpha\) for some \(t\). Since \(f(t)\) is an injective function, we can find \(t\) as \(t = f^{-1}(\frac{y}{x})\). Using the standard change of variables formulas from rectangular to polar coordinates

\[
x^2 + y^2 = r^2 \quad \quad \quad (2)
\]
\[
x = r \cos \theta \quad \quad \quad \quad \quad \quad (3)
\]
\[
y = r \sin \theta \quad \quad \quad \quad \quad \quad (4)
\]

We can rewrite equation 2 as:

\[
h^2 (f^{-1}(\frac{y}{x})) + g^2 (f^{-1}(\frac{y}{x})) = r^2 = x^2 + y^2
\]

In the following examples, theorem 1 is used to convert some parametric curves to implicit equivalents.

Example 1: Let \(h(t) = \frac{3t}{1+t^3}\) and \(g(t) = \frac{3t^2}{1+t^3}\)

\[
f(t) = t \quad \quad \quad t = \frac{y}{x}
\]

The corresponding implicit form will be:

\[
x^2 + y^2 = \left(\frac{3y}{x}\right)^2 + \left(\frac{3y^2}{x(1+y^3)}\right)^2
\]

Simplifying this equation gives \(x^3 + y^3 - 3xy = 0\)

The parametric curve for this example is plotted in Figure 1. The corresponding implicit form obtained using theorem 1 is plotted in Figure 2.

![Figure 1](image1.png)

![Figure 2](image2.png)

Example 2: Let \(h(t) = \sin t\) and \(g(t) = \sin t \cos t\)

\[
f(t) = \cos t \quad \quad \quad t = \cos^{-1}(\frac{y}{x})
\]

The corresponding implicit form will be:

\[
x^2 + y^2 = \sin^2(\cos^{-1}(\frac{y}{x})) + \sin^2(\cos^{-1}(\frac{y}{x})) \cos^2(\cos^{-1}(\frac{y}{x}))
\]

Simplifying this equation will yield: \(x^4 - x^2 + y^2 = 0\)
The parametric curve for this example is plotted in Figure 3. The corresponding implicit form obtained using theorem 1 is plotted in Figure 4.

![Figure 3. Plot of the parametric curve of example 2.](image1)

![Figure 4. Plot of the implicit curve of example 2.](image2)

Example 3: Let \( h(t) = a \cos t \) and \( g(t) = b \sin t \)

\[
f(t) = \frac{b \tan t}{a} \quad \text{and} \quad t = \tan^{-1}\left(\frac{ay}{bx}\right)
\]

The corresponding implicit form will be:

\[
x^2 + y^2 = b^2 \sin^2 (\tan^{-1}\left(\frac{ay}{bx}\right)) + a^2 \cos^2 (\tan^{-1}\left(\frac{ay}{bx}\right))
\]

Simplifying this equation will give \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)

Let \( a=1 \) and \( b=2 \);

The parametric curve for this example is plotted in Figure 5. The corresponding implicit form obtained using theorem 1 is plotted in Figure 6.

![Figure 5. Plot of the parametric curve of example 3.](image3)

![Figure 6. Plot of the implicit curve of example 3.](image4)

2.2 Conversion in 3D

The method presented in the previous section is applied to curves in 3D in this section. The difference is that in 3D we will have two slope functions for a space curve. The same equalities will be used for these two slope functions. Using standard change of variable formulas from cartesian to spherical coordinate representations, implicitization is simply achieved. The following theorem gives mathematical basis for this conversion method.

**Theorem 3:** Let \( \alpha \) be a parametrized space curve in \( \mathbb{R}^n \) s.t. \( \alpha: (a,b) \to \mathbb{R}^n \) with \( \alpha(t) = (h(t), g(t), k(t)) \), where \( (a,b) \) is an open interval in \( \mathbb{R} \).
The implicit representation of this curve is given as:

\[ h^2 \left( f_1^{-1} \left( \frac{y}{x} \right) \right) + g^2 \left( f_1^{-1} \left( \frac{y}{x} \right) \right) + k^2 \left( f_2^{-1} \left( \frac{z}{\sqrt{x^2 + y^2}} \right) \right) = x^2 + y^2 + z^2 \]  

(5)

Where

\[ f_1(t) \text{ is an injective function of the form } f_1(t) = \frac{g(t)}{h(t)} \]

and \( f_2(t) \) is an injective function of the form \( f_2(t) = \frac{k(t)}{\sqrt{g^2(t) + h^2(t)}} \)

**Proof:** Let \((r, \theta, \phi)\) be a polar representation of a point \(p = (x, y, z) \in \mathbb{R}^3\) on the curve \(\alpha\). \(x = h(t), y = g(t), z = k(t)\) for some \(t\). Since \(f_1(t)\) and \(f_2(t)\) are injective functions, we can find \(t\) for \(x\) and \(y\) as \(t = f_1^{-1} \left( \frac{y}{x} \right)\) and for \(z\) and \(x, y\) as \(t = f_2^{-1} \left( \frac{z}{\sqrt{x^2 + y^2}} \right)\). Using the standard change of variables formulas from cartesian to spherical coordinates

\[ x^2 + y^2 + z^2 = R^2 \]  

(6)

\[ x = R \cos \theta \sin \phi \]  

(7)

\[ y = R \sin \theta \sin \phi \]  

(8)

\[ z = R \cos \phi \]  

(9)

We can rewrite equation 6 as:

\[ h^2 \left( f_1^{-1} \left( \frac{y}{x} \right) \right) + g^2 \left( f_1^{-1} \left( \frac{y}{x} \right) \right) + k^2 \left( f_2^{-1} \left( \frac{z}{\sqrt{x^2 + y^2}} \right) \right) = R^2 = x^2 + y^2 + z^2 \]

Example 4: If we have a space curve representing a spherical spiral in 3D given as \(\alpha(t) = (h(t), g(t), k(t))\).

\[ h(t) = \cos^2 t \quad g(t) = \cos t \sin t \quad k(t) = \sin t \]

\[ f_1(t) = \frac{\sin t}{\cos t} \quad t = \arctan \frac{y}{x} \]

\[ f_2(t) = \frac{\sin t}{\cos t} \quad t = \arctan \frac{y}{x} \]

\[ x^2 + y^2 + z^2 = \cos^2 t (\cos^2 t + \sin^2 t) + \sin^2 t \]

Which simplifies to \(x^2 + y^2 + z^2 = 1\)

Example 5: If we have a space curve, representing a cylindrical spiral in 3D given as \(\alpha(t) = (h(t), g(t), k(t))\).

\[ h(t) = a \cos t \quad g(t) = a \sin t \quad k(t) = t \]
$$f_1(t) = \frac{\sin t}{\cos t}, \quad t = \arctan \frac{y}{x}$$

$$f_2(t) = \frac{t}{a}, \quad t = \frac{az}{\sqrt{x^2 + y^2}}$$

$$x^2 + y^2 + z^2 = a^2 (\cos^2 t + \sin^2 t) + t^2$$

Which simplifies to  

$$x^2 + y^2 = a^2$$

3. Conversion from Implicit Form to Parametric Form

In this section, a new method is introduced to convert an implicit form to a corresponding parametric form. This method basically depends on rewriting implicit polynomial in polar form for 2D and spherical form in 3D and solving the radius in terms of angles. For this reason the proposed method is applicable to any degree of polynomials, where roots of the polynomial can be explicitly obtained in algebraic form. In this paper, implicit forms up to sum of five successive homogeneous polynomials are given.

3.1 Conversion in 2D

In this section, conversion from implicit form to parametric form is studied. The conversion is based on using the polar coordinate representation of the implicit polynomial.

**Theorem 4:** A polynomial of the form

$$\sum_{m=0}^{n} a_m x^m y^{(a-m)} + \sum_{m=0}^{k} b_m x^m y^{(k-m)} = 0 \quad (10)$$

(sum of two homogeneous polynomials) can be converted to parametric form

$$\alpha(\theta) = (h(\theta), g(\theta))$$

as:

$$h(\theta) = \frac{\sum_{m=0}^{k} b_m \cos^m \theta \sin^{(k-m)} \theta}{\sum_{m=0}^{n} a_m \cos^m \theta \sin^{(n-m)} \theta} \quad \cos \theta \quad \text{(11)}$$

$$g(\theta) = \frac{\sum_{m=0}^{k} b_m \cos^m \theta \sin^{(k-m)} \theta}{\sum_{m=0}^{n} a_m \cos^m \theta \sin^{(n-m)} \theta} \quad \sin \theta \quad \text{(12)}$$

Parameter values to obtain real valued parametric form, specifies parameter range.

**Proof:** Equation 10 can be represented in polar coordinates as:

$$r^n \sum_{m=0}^{n} a_m \cos^m \theta \sin^{(n-m)} \theta + r^k \sum_{m=0}^{k} b_m \cos^m \theta \sin^{(k-m)} \theta = 0$$

If we solve the above equation for $r$, we obtain:
Using the polar to rectangular coordinate change of variable formulas 
\( \alpha(\theta) = (r \cos \theta, r \sin \theta) \) is obtained.

In the following examples, implicit forms obtained in section 2.1 are considered. To show that converted forms correspond to the same parametric curves, suitable reparametrizations are used for each example.

Example 6: Consider the implicit curve \( x^3 + y^3 - 3xy = 0 \)

Substituting the polar to rectangular coordinate conversion formulas for \( x \) and \( y \), and solving for \( r \), we obtain

\[
(r \cos \theta)^3 + (r \sin \theta)^3 - 3(r \cos \theta)(r \sin \theta) = 0
\]

\[
r = \frac{3 \cos \theta \sin \theta}{\cos^3 \theta + \sin^3 \theta}
\]

\[
h(\theta) = \frac{3 \cos^2 \theta \sin \theta}{\cos^3 \theta + \sin^3 \theta}, \quad g(\theta) = \frac{3 \cos \theta \sin^2 \theta}{\cos^3 \theta + \sin^3 \theta}, \quad \theta \in [0, 2\pi]
\]

If we reparametrize this equation using \( t = \tan \theta \)

We obtain

\[
h(t) = \frac{3t}{1 + t^2}, \quad g(t) = \frac{3t^2}{1 + t^2}
\]

which is the same parametric form given in Example 1.

Example 7: Consider the eight curve \( x^4 - x^2 + y^2 = 0 \)

\[
(r \cos \theta)^4 - (r \cos \theta)^2 + (r \sin \theta)^2 = 0
\]

\[
r^2 = \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}
\]

\[
h(\theta) = \frac{\sqrt{\cos^2 \theta - \sin^2 \theta}}{\cos \theta}, \quad g(\theta) = \frac{\sqrt{\cos^2 \theta - \sin^2 \theta}}{\cos \theta} \cdot \frac{\sin \theta}{\cos \theta}, \quad \theta \in [0, 2\pi]
\]

If we reparametrize this form using \( \cos t = \frac{\sin \theta}{\cos \theta} \)

We obtain \( h(t) = \sin t, g(t) = \sin t \cos t \)

which is the parametric form of eight curve.

Example 8: Consider the equation of an ellipse, \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)
\[ r^2 (b^2 \cos^2 \theta + a^2 \sin^2 \theta) = a^2 b^2 \]

\[ h(\theta) = \sqrt{\frac{a^2 b^2 \cos^2 \theta}{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \quad g(\theta) = \sqrt{\frac{a^2 b^2 \sin^2 \theta}{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \quad \theta \in [0, 2\pi] \]

If we reparametrize this function using

\[ \cos t = \frac{b \cos \theta}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \]

We obtain \( h(t) = a \cos t \quad g(t) = b \sin t \)

which is the parametric form of ellipse.

**Theorem 5:** A polynomial of the form

\[ \sum_{l=0}^{n-1} \sum_{m=0}^{n-l} a_{lm} x^l y^{(n-l-m)} = 0 \quad (14) \]

(three homogeneous polynomials of consecutive degrees) can be converted to two parametric forms as:

\( \alpha_1(t) = (h_1(t), g_1(t)) \quad (15) \quad \alpha_2(t) = (h_2(t), g_2(t)) \quad (16) \)

\[ h_1(\theta) = \frac{-p - \sqrt{p^2 - 4sq}}{2s} \cos \theta \quad (17) \quad g_1(\theta) = \frac{-p - \sqrt{p^2 - 4sq}}{2s} \sin \theta \quad (18) \]

\[ h_2(\theta) = \frac{-p + \sqrt{p^2 - 4sq}}{2s} \cos \theta \quad (19) \quad g_2(\theta) = \frac{-p + \sqrt{p^2 - 4sq}}{2s} \sin \theta \quad (20) \]

Where

\[ s = \sum_{m=0}^{n} a_{m0} \cos^m \theta \sin^{(n-m)} \theta \quad (21) \quad p = \sum_{m=0}^{n-1} a_{m1} \cos^m \theta \sin^{(n-1-m)} \theta \quad (22) \]

\[ q = \sum_{m=0}^{n-2} a_{m2} \cos^m \theta \sin^{(n-2-m)} \theta \quad (23) \]

Parameter values to obtain real valued parametric form, specifies parameter range.

**Proof:** Equation 14 can be represented in polar coordinates as:

\[ \sum_{l=0}^{n-1} \sum_{m=0}^{n-l} r^{n-l} a_{lm} \cos^m \theta \sin^{(n-l-m)} \theta = 0 \]

\[ r^{n-2} \sum_{l=0}^{n-2} \sum_{m=0}^{n-l} a_{ml} \cos^m \theta \sin^{(n-l-m)} \theta = 0 \quad (24) \]

\[ sr^2 + pr + q = 0 \quad (25) \]

Solving this quadratic equation in terms of \( r \) and using polar to rectangular coordinate conversion formulas will give parametric representations of the polynomial given in equation 15 and 16. Closed form solution of this quadratic equation can be found in the reference [1].
Example 9: \( x^2 + y^2 - 2x - 2y + 1 = 0 \)
\[ r^2 - 2r(\cos \theta + \sin \theta) + 1 = 0 \]

To find real valued roots of this equation, the following constraint has to be satisfied
\[ 4(\cos \theta + \sin \theta)^2 - 4 \geq 0 \] which is satisfied for \( \theta \in \left[ 0, \frac{\pi}{2} \right] \)

The solutions of the above equation are:
\[ r_1 = \cos \theta + \sin \theta - \sqrt{\sin 2\theta} \quad r_2 = \cos \theta + \sin \theta + \sqrt{\sin 2\theta} \]

and the corresponding two parametric curves are obtained as:
\[ h_1(\theta) = r_1 \cos \theta \quad g_1(\theta) = r_1 \sin \theta \]
\[ h_2(\theta) = r_2 \cos \theta \quad g_2(\theta) = r_2 \sin \theta \]
\[ \theta \in \left[ 0, \frac{\pi}{2} \right] \]

The implicit polynomial form of the curve given in this example is plotted in Figure 7. Corresponding two parametric forms obtained from theorem 5 are plotted in Figure 8 and Figure 9.

Example 10: Consider the implicit curve, \( x^2 + xy + 4y^2 + 5x + 2y - 100 = 0 \)
\[ r^2 (1 + \cos \theta \sin \theta + 3 \sin^2 \theta) + r(5 \cos \theta + 2 \sin \theta) - 100 = 0 \]

To find real valued roots of this equation, the following constraint has to be satisfied
\[ 1179\sin^2 \theta + 210\sin 2\theta + 425 > 0 \] which is satisfied \( \forall \theta \)

\[ r_1 = \frac{- (5 \cos \theta + 2 \sin \theta) - \sqrt{1179\sin^2 \theta + 210\sin 2\theta + 425}}{2(1 + \cos \theta \sin \theta + 3 \sin^2 \theta)} \]
\[ r_2 = \frac{- (5 \cos \theta + 2 \sin \theta) + \sqrt{1179 \sin^2 \theta + 210 \sin 2\theta + 425}}{2(1 + \cos \theta \sin \theta + 3 \sin^2 \theta)} \]

\[
h_1(\theta) = r_1 \cos \theta \quad g_1(\theta) = r_1 \sin \theta
\]

\[
h_2(\theta) = r_2 \cos \theta \quad g_2(\theta) = r_2 \sin \theta \quad \theta \in [0, 2\pi]
\]

The implicit polynomial form of the curve given in this example is plotted in Figure 10. For this example two parametric curves obtained are the same. Corresponding parametric form obtained from theorem 5 is plotted in Figure 11.

Theorem 6: A polynomial of the form

\[
\sum_{l=0}^{3} \sum_{m=0}^{n-l} a_{lm} x^m y^{(n-l-m)} = 0
\]  

(26)

(four homogeneous polynomials of consecutive degrees) can be converted to three parametric forms as:

\[
\alpha_1(t) = (h_1(t), g_1(t))
\]  

(27)

\[
\alpha_2(t) = (h_2(t), g_2(t))
\]  

(28)

\[
\alpha_3(t) = (h_3(t), g_3(t))
\]  

(29)

\[
h_1(\theta) = (A + B) \cos \theta
\]  

(30)

\[
g_1(\theta) = (A + B) \sin \theta
\]  

(31)

\[
h_2(\theta) = (-\frac{A + B}{2} + \frac{A - B}{2} \sqrt{3}) \cos \theta
\]  

(32)

\[
g_2(\theta) = (-\frac{A + B}{2} + \frac{A - B}{2} \sqrt{3}) \sin \theta
\]  

(33)

\[
h_3(\theta) = (-\frac{A + B}{2} - \frac{A - B}{2} \sqrt{3}) \cos \theta
\]  

(34)

\[
g_3(\theta) = (-\frac{A + B}{2} - \frac{A - B}{2} \sqrt{3}) \sin \theta
\]  

(35)

where

\[
A = \sqrt{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \quad (36) \quad B = \sqrt{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}
\]  

(37)
\[ a = \frac{1}{3} \left( 3 \frac{q}{s} - \frac{p^2}{s^2} \right) \]  \quad (38) \quad b = \frac{1}{27} \left( 2 \frac{p^3}{s^3} - 9 \frac{pq}{s^2} + 27 \frac{u}{s} \right) \]  \quad (39)

\( s, p, q \) are defined in equations 21 to 23.

\[ u = \sum_{m=0}^{n-3} a_{m3} \cos^m \theta \sin^{(n-3-m)} \theta \]  \quad (40)

Parameter values to obtain real valued parametric form, specifies parameter range.

**Proof:** Equation 26 can be represented in polar coordinates as:

\[ \sum_{l=0}^{3} \sum_{m=0}^{n-3-l} r^{n-3-l} a_{ml} \cos^m \theta \sin^{(n-3-m)} \theta = 0 \]  \quad (41)

\[ sr^3 + pr^2 + qr + u = 0 \]  \quad (42)

Solving this cubic equation in terms of \( r \) and using polar to rectangular coordinate conversion formulas will give parametric representations of the polynomial given in equation 27 to 29. Closed form solution of this cubic equation can be found in the reference [1].

**Theorem 7:** A polynomial of the form  
\[ \sum_{l=0}^{4} \sum_{m=0}^{n-4-l} a_{ml} x^m y^{(n-4-l)} = 0 \]  \quad (43)

(five homogeneous polynomials of consecutive degrees) can be converted to four parametric forms as:

\[ \alpha_1(t) = (h_1(t), g_1(t)) \]  \quad (44) \quad \alpha_2(t) = (h_2(t), g_2(t)) \]  \quad (45) \quad \alpha_3(t) = (h_3(t), g_3(t)) \]  \quad (46) \quad \alpha_4(t) = (h_4(t), g_4(t)) \]  \quad (47)

\[ h_1(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} + \frac{D}{2} \right) \cos \theta \]  \quad (48) \quad g_1(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} + \frac{D}{2} \right) \sin \theta \]  \quad (49)

\[ h_2(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} - \frac{D}{2} \right) \cos \theta \]  \quad (50) \quad g_2(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} - \frac{D}{2} \right) \sin \theta \]  \quad (51)

\[ h_3(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} + \frac{E}{2} \right) \cos \theta \]  \quad (52) \quad g_3(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} + \frac{E}{2} \right) \sin \theta \]  \quad (53)

\[ h_4(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} - \frac{E}{2} \right) \cos \theta \]  \quad (54) \quad g_4(\theta) = \left( -\frac{p}{4s} + \frac{R}{2} - \frac{E}{2} \right) \sin \theta \]  \quad (55)

Let \( l \) be any root of the equation  
\[ l^3 - \frac{q}{s} l^2 + \left( \frac{pu}{s^2} - 4v \right) l - \frac{p^2 v}{s^3} + 4 \frac{q v}{s^2} - \frac{u^2}{s^2} = 0 \]  \quad (56)
\[ R = \sqrt{\frac{p - q + l}{s}} \]  
(57)

\[ D = \sqrt{\frac{3p^2 - R^2 - 2q + 4pq - 8us^2 - p^3}{4s^3 R}} \]  
(58)

\[ E = \sqrt{\frac{3p^2 - R^2 - 2q - 4pq - 8us^2 - p^3}{4s^3 R}} \]  
(59)

\[ s, p, q \] are defined in equations 21 to 23. \( u \) is defined in equation 40.

\[ v = \sum_{m=0}^{n-4} a_m \cos^m \theta \sin^{(n-4-m)} \theta \]  
(60)

Parameter values to obtain real valued parametric form, specifies parameter range.

**Proof:** Equation 43 can be represented in polar coordinates as:

\[ \sum_{l=0}^{4} \sum_{m=0}^{n-l} r^{n-l} a_{ml} \cos^m \theta \sin^{(n-l-m)} \theta = 0 \]

\[ r^{n-4} \sum_{l=0}^{4} \sum_{m=0}^{n-l} a_{ml} \cos^m \theta \sin^{(n-l-m)} \theta = 0 \]  
(61)

\[ sr^4 + pr^3 + qr^2 + ur + v = 0 \]  
(62)

Solving this quartic equation in terms of \( r \) and using polar to rectangular coordinate conversion formulas will give parametric representations of the polynomial given in equations 44 to 47. Solution of this quadratic equation can be found in the reference [1].

**3.2 Conversion in 3D**

In this section we study the problem of conversion from implicit form to parametric form in 3D using spherical coordinate representation of the implicit polynomial. As in the previous section, the method depends on finding roots of this form by taking radius as a variable and sine, cosine terms as coefficients. Patches are obtained from this implicit to parametric conversion in 3D. In this section, proofs are not given but referenced to the previous section, since similar derivations can be carried out in both cases.

**Theorem 8:** A polynomial of the form

\[ \sum_{m=0}^{n} \left[ d_m x^m y^{(n-m)} + b_m x^m z^{(n-m)} + c_m y^m z^{(n-m)} \right] + \sum_{m=1}^{n} \sum_{t=1}^{n} \left[ d_{mt} x^m y^t z^{(n-m-t)} \right] \]

\[ + \sum_{m=0}^{k} \left[ e_m x^m y^{(k-m)} + f_m x^m z^{(k-m)} + g_m y^m z^{(k-m)} \right] + \sum_{m=1}^{k} \sum_{t=1}^{k} \left[ h_{mt} x^m y^t z^{(k-m-t)} \right] = 0 \]  
(63)

can be written as a patch of the form:
\[
x(\theta, \varphi) = \\
\sqrt{\sum_{m=0}^{n-k} \left[ a_m \alpha^m \beta^{(k-m)} + b_m \alpha^m \gamma^{(k-m)} + c_m \beta^m \gamma^{(k-m)} \right] + \sum_{m=1}^{n-k} \sum_{i=1}^{k} \left[ h_{m,i} \alpha^m \beta^i \gamma^{(k-m-i)} \right] + \sum_{m=1}^{n-k} \sum_{i=1}^{k} \left[ d_{m,i} \alpha^m \beta^i \gamma^{(k-m-i)} \right]}
\]
\[
y(\theta, \varphi) = \\
\sqrt{\sum_{m=0}^{n-k} \left[ a_m \alpha^m \beta^{(k-m)} + b_m \alpha^m \gamma^{(k-m)} + c_m \beta^m \gamma^{(k-m)} \right] + \sum_{m=1}^{n-k} \sum_{i=1}^{k} \left[ h_{m,i} \alpha^m \beta^i \gamma^{(k-m-i)} \right] + \sum_{m=1}^{n-k} \sum_{i=1}^{k} \left[ d_{m,i} \alpha^m \beta^i \gamma^{(k-m-i)} \right]}
\]
\[
z(\theta, \varphi) = \\
\sqrt{\sum_{m=0}^{n-k} \left[ a_m \alpha^m \beta^{(k-m)} + b_m \alpha^m \gamma^{(k-m)} + c_m \beta^m \gamma^{(k-m)} \right] + \sum_{m=1}^{n-k} \sum_{i=1}^{k} \left[ h_{m,i} \alpha^m \beta^i \gamma^{(k-m-i)} \right] + \sum_{m=1}^{n-k} \sum_{i=1}^{k} \left[ d_{m,i} \alpha^m \beta^i \gamma^{(k-m-i)} \right]}
\]

where
\[
\alpha = \sin \varphi \cos \theta \quad (67) \quad \beta = \sin \varphi \sin \theta \quad (68) \quad \gamma = \cos \varphi \quad (69)
\]

Parameter values to obtain real valued patch, specifies parameter ranges.

**Proof:** The proof of this theorem is similar to the proof of Theorem 4.

Example 11: If we have a space curve, representing a spherical spiral given in example 4. By converting this spiral with implicit polynomials we obtained \( x^2 + y^2 + z^2 = 1 \).

From conversion formulas given in theorem 8, we obtain a patch of the form:

\[
x(\theta, \varphi) = \cos \theta \sin \varphi \quad y(\theta, \varphi) = \sin \theta \sin \varphi \quad z(\theta, \varphi) = \cos \varphi \quad \theta \in [0,2\pi], \ \varphi \in [0,2\pi]
\]

Example 12: If we have a space curve, representing a cylindrical spiral given in example 5. By converting this spiral with implicit polynomials we obtained \( x^2 + y^2 = a^2 \).

From conversion formulas given in theorem 8, we obtain a patch of the form:

\[
x(\theta, \varphi) = a \cos \theta \quad y(\theta, \varphi) = a \sin \theta \quad z(\theta, \varphi) = a \tan \varphi \quad \theta \in [0,2\pi], \ \varphi \in [0,2\pi]
\]
Theorem 9: A polynomial of the form
\[
\sum_{l=0}^{2} \left[ \sum_{n=0}^{n-l-1} a_{ml} x^m y^{n-l-m} + b_{ml} x^m z^{n-l-m} + c_{ml} y^m z^{n-l-m} \right] + \sum_{m=1}^{n-l} \sum_{r=1}^{n-l} d_{mr} x^m y^r z^{n-l-m-r} = 0
\]  
(70)
can be written as two patches of forms:
\[
x_1(\theta, \phi) = \frac{-p + \sqrt{p^2 - 4sq}}{2s} \sin \phi \cos \theta
\]  
(71)  
\[
y_1(\theta, \phi) = \frac{-p + \sqrt{p^2 - 4sq}}{2s} \sin \phi \sin \theta
\]  
(72)  
\[
z_1(\theta, \phi) = \frac{-p + \sqrt{p^2 - 4sq}}{2s} \cos \phi
\]  
(73)
\[
x_2(\theta, \phi) = \frac{-p - \sqrt{p^2 - 4sq}}{2s} \sin \phi \cos \theta
\]  
(74)  
\[
y_2(\theta, \phi) = \frac{-p - \sqrt{p^2 - 4sq}}{2s} \sin \phi \sin \theta
\]  
(75)  
\[
z_2(\theta, \phi) = \frac{-p - \sqrt{p^2 - 4sq}}{2s} \cos \phi
\]  
(76)
\[
s = \sum_{m=0}^{n} \left[ a_{m0} \alpha^m \beta^m \gamma^{n-m} \right] + \sum_{m=1}^{n} \sum_{r=1}^{n} \left[ d_{mr} \alpha^m \beta^r \gamma^{n-m-r} \right] \]  
(77)  
\[
p = \sum_{m=0}^{n} \left[ a_{m1} \alpha^m \beta^m \gamma^{n-m-1} \right] + \sum_{m=1}^{n} \sum_{r=1}^{n} \left[ d_{mr} \alpha^m \beta^r \gamma^{n-m-1-r} \right] \]  
(78)  
\[
q = \sum_{m=0}^{n} \left[ a_{m2} \alpha^m \beta^m \gamma^{n-2-m} \right] + \sum_{m=1}^{n} \sum_{r=1}^{n} \left[ d_{mr} \alpha^m \beta^r \gamma^{n-2-m-r} \right] \]  
(79)
\[
\alpha, \beta, \gamma \text{ are defined in equations } 67 \text{ to } 69.
\]

Parameter values to obtain real valued patches, specifies parameter ranges.

Proof: The proof of this theorem is similar to the proof of Theorem 5.

Theorem 10: A polynomial of the form
\[
\sum_{l=0}^{3} \left[ \sum_{n=0}^{n-l-1} a_{ml} x^m y^{n-l-m} + b_{ml} x^m z^{n-l-m} + c_{ml} y^m z^{n-l-m} \right] + \sum_{m=1}^{n-l} \sum_{r=1}^{n-l} d_{mr} x^m y^r z^{n-l-m-r} = 0
\]  
(80)
can be written as three patches of forms:
\[
x_1(\theta, \phi) = (A + B) \sin \phi \cos \theta
\]  
(81)  
\[
y_1(\theta, \phi) = (A + B) \sin \phi \sin \theta
\]  
(82)
where
\[ A = \sqrt{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}, \quad B = \sqrt{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \]
\[ a = \frac{1}{3}(s^2 - \frac{p^2}{s^2}), \quad b = \frac{1}{27}(2s^3 - 9pq + 27u) \]

\( s, p, q \) are defined in equations 77 to 79.

\[ u = \sum_{m=0}^{n-1} \sum_{m=0}^{n-1} \left[ a_{m} \alpha^{m} y^{(n-3-m)} + b_{m} \alpha^{m} y^{(n-3-m)} + c_{m} \beta^{m} y^{(n-3-m)} \right] + \sum_{m=-1}^{n-1} \sum_{m=1}^{n-1} \left[ a_{m} \alpha^{m} \beta^{m} y^{(n-3-m-1)} \right] \]
\[ \alpha, \beta, \gamma \] are defined in equations 67 to 69.

Parameter values to obtain real valued patches, specifies parameter ranges.

**Proof:** The proof of this theorem is similar to the proof of Theorem 6.

**Theorem 11:** A polynomial of the form
\[ \sum_{i=0}^{4} \sum_{m=0}^{n-1} \left[ a_{m} \alpha^{m} y^{(n-1-m)} + b_{m} \alpha^{m} y^{(n-1-m)} + c_{m} \beta^{m} y^{(n-1-m)} \right] + \sum_{m=-1}^{n-1} \sum_{m=1}^{n-1} \left[ a_{m} \alpha^{m} \beta^{m} y^{(n-1-m)} \right] = 0 \]
can be written as four patches of forms:
\[ x_{1}(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} + \frac{D}{2}) \sin \varphi \cos \theta \]
\[ y_{1}(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} + \frac{D}{2}) \sin \varphi \sin \theta \]
\[ z_1(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} + \frac{D}{2}) \cos \varphi \]  
(98)

\[ x_2(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} - \frac{D}{2}) \sin \varphi \cos \theta \]  
(99)

\[ y_2(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} - \frac{D}{2}) \sin \varphi \sin \theta \]  
(100)

\[ z_2(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} - \frac{D}{2}) \cos \varphi \]  
(101)

\[ x_3(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} + \frac{E}{2}) \sin \varphi \cos \theta \]  
(102)

\[ y_3(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} + \frac{E}{2}) \sin \varphi \sin \theta \]  
(103)

\[ z_3(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} + \frac{E}{2}) \cos \varphi \]  
(104)

\[ x_4(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} - \frac{E}{2}) \sin \varphi \cos \theta \]  
(105)

\[ y_4(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} - \frac{E}{2}) \sin \varphi \sin \theta \]  
(106)

\[ z_4(\theta, \varphi) = (-\frac{p}{4s} + \frac{R}{2} - \frac{E}{2}) \cos \varphi \]  
(107)

Let \( o \) be any root of the equation \( o^3 - \frac{q}{s} u^2 + \left(\frac{pu}{s^2} - 4v\right) o^2 - \frac{p^2 v}{s^3} + 4 \frac{qv}{s^2} - \frac{u^2}{s^2} = 0 \)  
(108)

\[ R = \sqrt{\frac{p}{4s} - \frac{q}{s} + o} \]  
(109)

\[ D = \sqrt{\frac{3p^2}{4s^2} - R^2 - 2 \frac{q}{s} + \frac{4pq}{s} - 8us^2 - p^3} \]  
(110)

\[ E = \sqrt{\frac{3p^2}{4s^2} - R^2 - 2 \frac{q}{s} - \frac{4pq}{s} - 8us^2 - p^3} \]  
(111)

\( s, p, q \) are defined in equations 77 to 79. \( u \) is defined in equation 94.

\[ v = \sum_{m=0}^{n-4} [a_m \alpha^m \gamma^{(n-4-m)} + b_m \alpha^m \gamma^{(n-4-m)} + c_m \beta^m \gamma^{(n-4-m)}] + \sum_{m=1}^{n-4} \sum_{l=1}^{n-4} [d_{m+l} \alpha^m \beta^l \gamma^{(n-4-m-l)}] \]  
(112)

\( \alpha, \beta, \gamma \) are defined in equations 67 to 69.

Proof: The proof of this theorem is similar to the proof of Theorem 7.
4. Conclusions

In this paper, conversion methods between parametric implicit forms of curves in 2D and 3D are considered. The conversion between these two forms is important because each form has advantages and disadvantages. Having a conversion formula between these two forms allow us to use advantages of both forms at the same time.

A new method is introduced to convert a parametric form of a curve to the corresponding implicit form. The method depends on finding slope function of the curve in parametric form and implicit form and equating them and using standard change of variable formulas from cartesian to polar/spherical coordinates. The strength of this method over existing methods is its simplicity.

Conversion from implicit form to a corresponding parametric form is also introduced. This conversion method also depends on polar/spherical coordinate representations and finding roots of radius function. The method is applicable to any degree of polynomial that roots of the polynomial can be obtained explicitly.

Conversion formulas introduced in this paper are applicable for broad range of functions in parametric form and implicit form. One main advantage of these two conversion methods is that, their ease of usage.

References